The Laplace Transform

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Definition Integral Transform

An improper integral of the form

$$\int_{-\infty}^{\infty} K(s,t)\; F(t)\; dt$$

is Called integral transform of $F(t)$ if it is Convergent. Sometimes it is denoted by $f(s)$ or $T\{F(t)\}$.

Thus

$$f(s) = T\{F(t)\} = \int_{-\infty}^{\infty} K(s,t)\; F(t)\, dt \quad -(1)$$

The function $K(s,t)$ appearing in the integrand is Called Kernel of the transform. Here s is a parameter and is independent of t, s may be real or Complex number.

If we take $K(s,t) = \begin{cases} e^{-st}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

Then (1) becomes

$$f(s) = T\{F(t)\} = \int_{0}^{\infty} F(t)\, e^{-st}\, dt$$

This transform is known as Laplace transform

(ii) Hankel Transform of $F(t)$ is

$$f(s) = \int_{0}^{\infty} F(t) \cdot t \cdot J_n(st)\, dt$$

(iii) Fourier Transform of $F(t)$ is

$$f(s) = \int_{-\infty}^{\infty} F(t)\, e^{-ist}\, dt.$$

Laplace Transform:

Suppose $F(t)$ is a real valued function defined over the interval $(-\infty, \infty)$ s.t $F(t) = 0 \ \forall \ t < 0$

The Laplace transform of $F(t)$ denoted by $L\{F(t)\}$, is defined as

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt \quad \text{--- (1)}$$

We also write

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Here L is called Laplace Transformation operator.
The parameter s is a real or Complex number.

In general, the Parameter s is taken to be a real positive number.

The Laplace transform is said to exist if the integral (1) is cgt for some value of s.

Theorem 1 **Linear property**

Suppose $f_1(s)$ and $f_2(s)$ are Laplace forms of $F_1(t)$ and $F_2(t)$ respectively

Then

$$L\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

where c_1 and c_2 are any Constants.

proof L.H.S

$$L\{c_1 F_1(t) + c_2 F_2(t)\}$$

$$= \int_0^\infty e^{-st}\left[c_1 F_1(t) + c_2 F_2(t)\right] dt$$

$$= c_1 \int_0^\infty e^{-st} F_1(t)\, dt + c_2 \int_0^\infty e^{-st} F_2(t)\, dt$$

$$= c_1 L\{F_1(t)\} + c_2 L\{F_2(t)\}$$

proved

Generalising this result, we obtain

$$L\left\{\sum_{r=1}^{n} c_r F_r(t)\right\} = \sum_{r=1}^{n} c_r L\{F_r(t)\}$$

Theorem 2 **First Shifting Theorem** (First Translation)

If $L\{F(t)\} = f(s)$ then

$$L\{e^{at} F(t)\} = f(s-a)$$

proof ~~If $L\{F(t)\} = f(s)$, then~~

Let $L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$

Then $L\{e^{at} F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\,dt$

$$= \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$= \int_0^\infty e^{-ut} F(t)\,dt \quad \text{where}$$

$u = s-a > 0$,

$$= f(u) = f(s-a)$$

proved

Theorem 3 Second Shifting Theorem
(Second Translation)

If $L\{F(t)\} = f(s)$ and

$$G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$$

Then $L\{G(t)\} = e^{-at} f(s)$

proof Let $L\{F(t)\} = f(s)$

$$G(t) = \begin{cases} F(t-a), & \text{if } t > a \\ 0, & \text{if } t < a \end{cases}$$

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= 0 + \int_a^\infty e^{-st} F(t-a)\,dt$$

put $t-a = p$ so that $dt = dp$

If $t = a$ then $p = t-a = a-a = 0$

If $t = \infty$ then $p = \infty - a = \infty$

$$\therefore L\,G(t) = \int_0^\infty e^{-s(p+a)} F(p)\,dp$$

$$= e^{-sa} \int_0^\infty e^{-sp} F(p)\,dp$$

$$= e^{-sa} f(s) \quad \text{proved}$$

Theorem 4 Change of Scale Property

If $L\{F(t)\} = f(s)$

Then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

proof

Let $L\, F(t) = f(s)$, then

$$L\{F(at)\} = \int_0^\infty e^{-st} F(at)\, dt$$

put $at = x$

$$= \int_0^\infty e^{-s x/a} F(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{sx}{a}} F(x)\, dx$$

$$= \frac{1}{a}\int_0^\infty e^{-s\frac{t}{a}} F(t)\, dt$$

$$= \frac{1}{a}\int_0^\infty e^{-pt} F(t)\, dt$$

$$= \frac{1}{a} f(p) \qquad p = s/a$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proved

Some Standard Results

(i) $\boxed{L(1) = \frac{1}{s}}$, $s > 0$

for $L(1) = \int_0^\infty e^{-st} \cdot 1 \, dt$

$$= \left[\frac{-e^{-st}}{s} \right]_0^\infty = \frac{1}{s} \text{ if } s>0$$

(ii) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$ if $s>0$, $n>-1$

For $L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$

Put $st = x$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) \text{ if } s>0, n+1>0$$

$\boxed{\because \Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx}$

In particular if n is a positive integer, then $\Gamma(n+1) = n!$ So that

$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$ if $s>0$ and n is a positive integer

(iii) $L\{e^{at}\} = \frac{1}{s-a}$, if $s > a$

<u>proof</u> $L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{-e^{-(s-a)t}}{s-a} \right]_{t=0}^{t=\infty}$$

$$= \frac{1}{s-a}, \text{ if } s > a$$

(iv) $L\{\cos at\} = \frac{s}{s^2+a^2}$, $L\{\sin at\} = \frac{a}{s^2+a^2}$

we have $L\{e^{at}\} = \frac{1}{s-a}$, $s > 0$

$$L\{e^{iat}\} = \frac{1}{s-ia} = \frac{s+ia}{(s-ia)(s+ia)}$$

$$= \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$L\{\cos at + i\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

$$\Rightarrow L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Equating real & imaginary parts we get-

$L\{\cos at\} = \frac{s}{s^2+a^2}$ &

$L\{\sin at\} = \frac{a}{s^2+a^2}$ Hence proved.

(v) $L\{\cosh at\} = \frac{s}{s^2-a^2}$ if $s > |a|$

& $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > |a|$

<u>proof</u> $L\{\cosh at\} = L\left\{\frac{e^{at}+e^{-at}}{2}\right\}$

$$= \frac{1}{2}L\{e^{at}\} + \frac{1}{2}L\{e^{-at}\}$$

By Linearity property.

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{1}{2}\left[\frac{s+\cancel{a}+s-\cancel{a}}{s^2-a^2}\right] = \frac{s}{s^2-a^2}, \ s > |a|$$

Similarly $L\{\sinh at\} = \frac{a}{s^2-a^2}$, $s > |a|$